Esquemas de actualización que preservan características dinámicas

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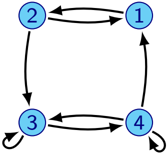
Contents

Motivation

Boolean Networks

- A finite set V of n element and n states variables $x_v \in \{0,1\}, \ v \in V$
- A global activation function $F = (f_v)_{v \in V} : \{0,1\}^n \to \{0,1\}^n$
  - Composed by local activation functions $f_v : \{0,1\}^n \to \{0,1\}$
- Schedule



$$\begin{aligned} f_1(x) &= x_2 \wedge x_4 \\ f_2(x) &= x_1 \\ f_3(x) &= x_2 \vee x_3 \\ f_4(x) &= x_3 \wedge x_4 \end{aligned}$$

$$s(i) = 1 \iff s = \{1,2,3,4\}$$

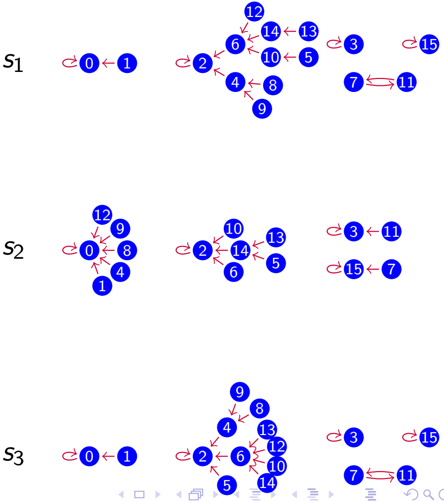
$$s(i) = i \iff s = \{1\}\{2\}\{3\}\{4\}$$

$$s(i) = \begin{cases} 1 & \text{if } i > 2 \\ 2 & \text{if } i \leq 2 \end{cases} \iff s = \{3,4\}\{1,2\}$$

Dynamics

$$s_1 = \{1,2,3,4\}$$
$$s_2 = \{1\}\{2\}\{3\}\{4\}$$
$$s_3 = \{3,4\}\{1,2\}$$

	State	s_1	s_2	s_3
0	0000	0000	0000	0000
1	0001	0000	0000	0000
2	0010	0010	0010	0010
3	0011	0011	0011	0011
4	0100	0010	0000	0010
5	0101	1010	1110	0010
6	0110	0010	0010	0010
7	0111	1011	1111	1011
8	1000	0100	0000	0100
9	1001	0100	0000	0100
10	1010	0110	0010	0110
11	1011	0111	0011	0111
12	1100	0110	0000	0110
13	1101	1110	1110	0110
14	1110	0110	0010	0110
15	1111	1111	1111	1111



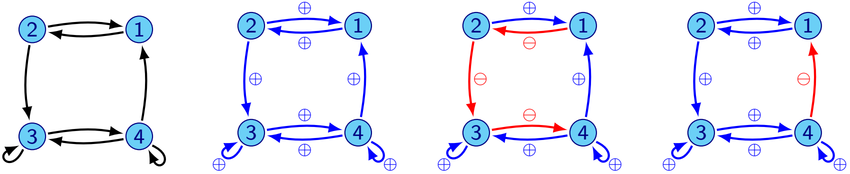
Labeled digraph

A labeled digraph is a graph G with a label function lab, (G, lab) such that: $lab : A(G) \to \{\oplus, \ominus\}$

We say that a labeled digraph is an update digraph if there exists $s : V(G) \to \{1, \dots, n\}$, an update function such that:

$$\forall (u, v) \in A(G), lab(u, v) = \oplus \iff s(u) \geq s(v)$$

$s_1 = \{1, 2, 3, 4\}$ $s_2 = \{1\}\{2\}\{3\}\{4\}$ $s_3 = \{3, 4\}\{1, 2\}$

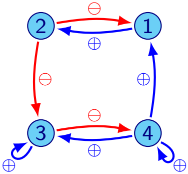


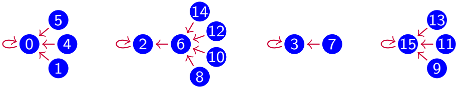
Why are we interested in labeled digraphs?

Theorem. (Aracena et al (2008))

Given two Boolean networks $N_1 = (F, s)$ and $N_2 = (F, s')$ which differ only in the update schedule. If the labeled digraphs associated to them are equal, then both networks have the same dynamical behavior.

$s = \{2\}\{1\}\{3\}\{4\}$ and $s' = \{2\}\{3\}\{1\}\{4\}$





Labels and Update digraphs

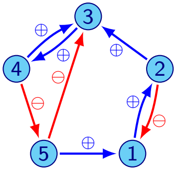
Forbidden cycle

A forbidden cycle is a cycle with an arc labeled $\ominus$ in the reverse graph.

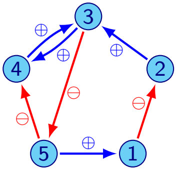
Theorem (Montalva (2012))

A labeled digraph is an update digraph if and only if there does not exist a forbidden cycle in its reverse digraph.

Labeled digraph



Reverse digraph



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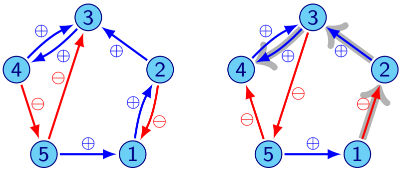

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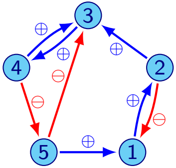

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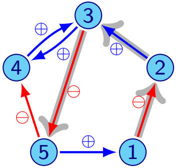

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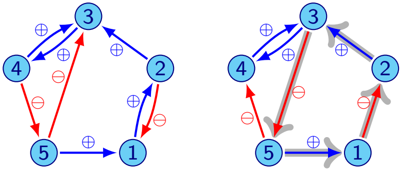

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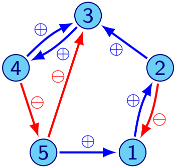

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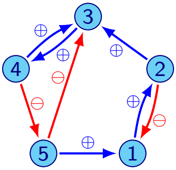

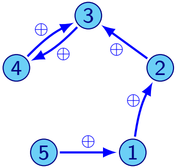

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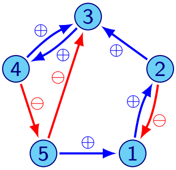

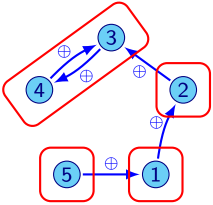

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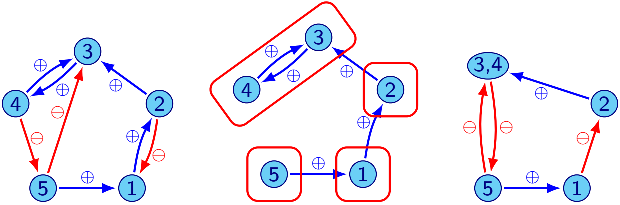
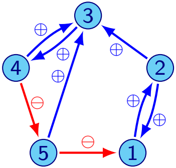


How I find the update schedule of a label?



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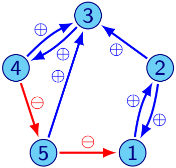

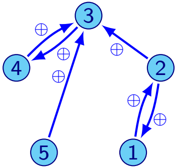

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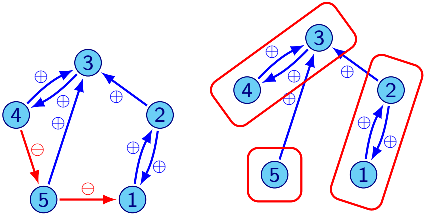

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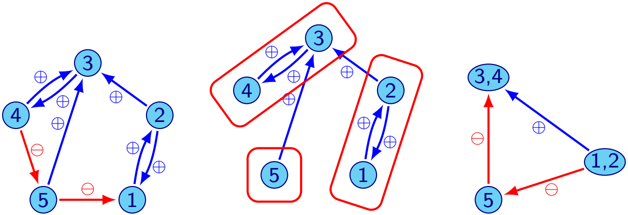

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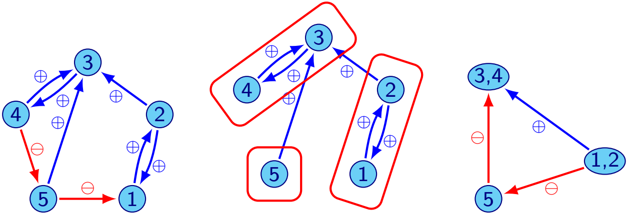

$$s = \{3,4\}\,\{5\}\,\{1,2\}$$

Update Digraph Extension Problem

UDE

Given a labeled digraph (G, lab), find the set $\mathcal{S}(G, lab)$ of all fully labeled extensions lab' of lab such that (G, lab') is an update digraph.

Complexity

CUDE

Given (G, lab) a labeled digraph, to determine the cardinality of the set $\mathcal{S}(G, lab)$.

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Given (G, lab) a labeled digraph, to determine the cardinality of the set $\mathcal{S}(G, lab)$.

Theorem

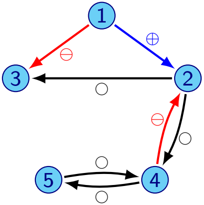
CUDE is #P-complete

Force

Proposition

Given a labeled digraph (G, lab) and an arc (i, j) with $lab(i, j) = \bigcirc$:

- If there exists a reverse path from i to j, then the arc (i, j) must be labeled $\oplus$
- If there exists a negative reverse path from j to i, then the arc (i, j) must be labeled $\ominus$



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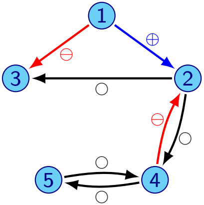

Matrix M

$V(G)$	1	2	3	4	5
1	∞	1	∞	-1	∞
2	∞	∞	∞	-1	∞
3	-1	-1	∞	-1	∞
4	∞	∞	∞	∞	∞
5	∞	∞	∞	∞	∞

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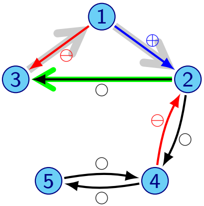

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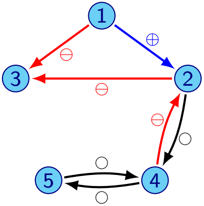

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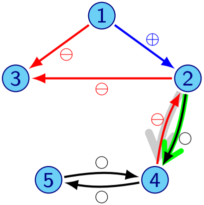

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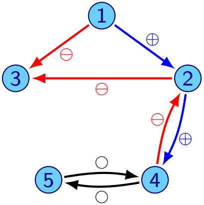

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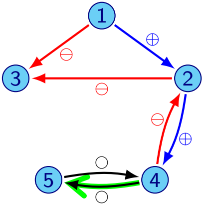

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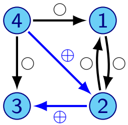
Example of SimpleLabel



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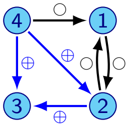

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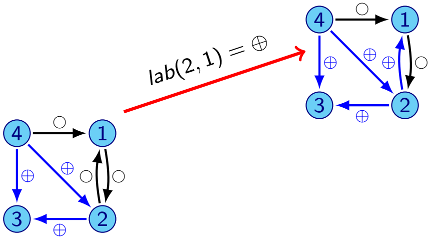

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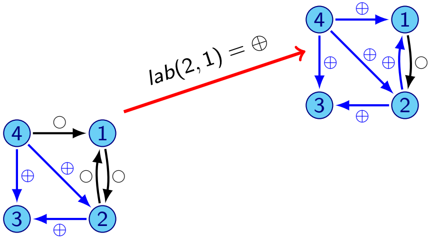

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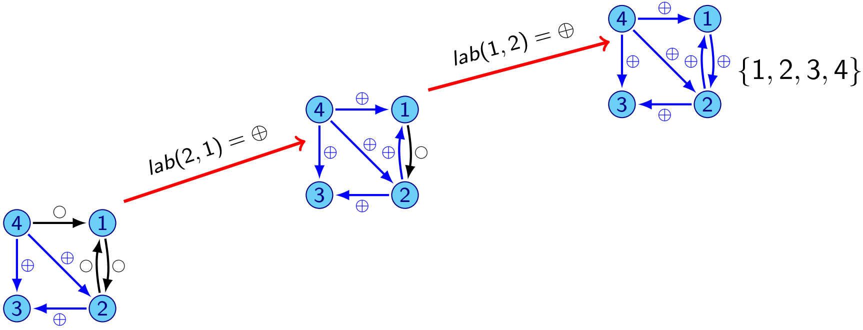

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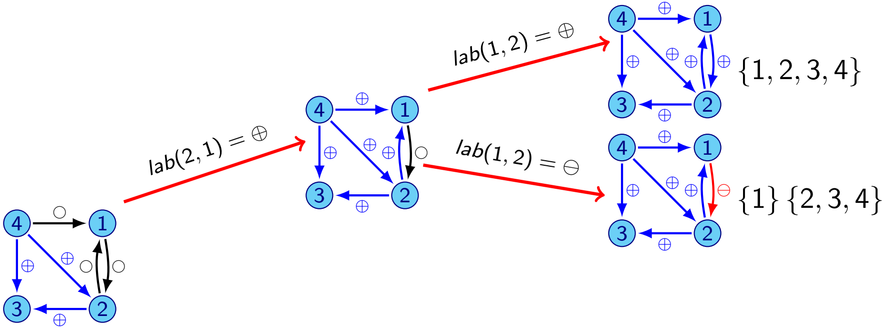

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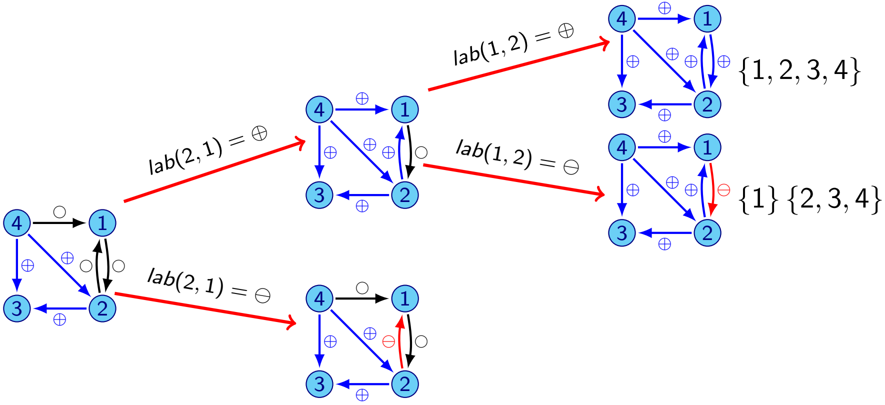

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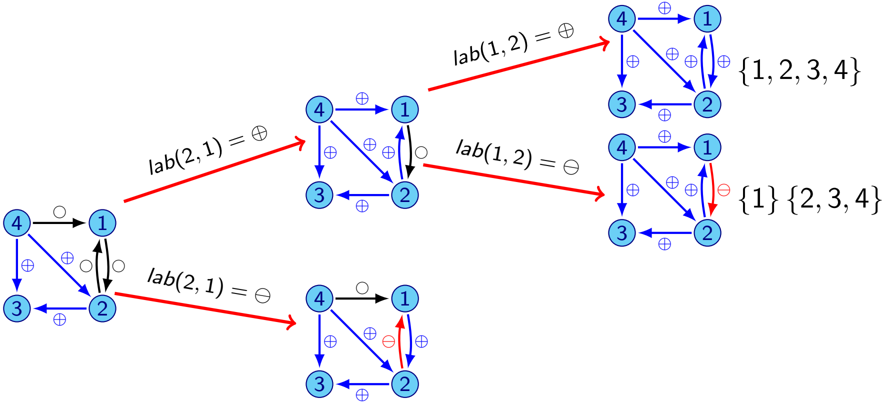

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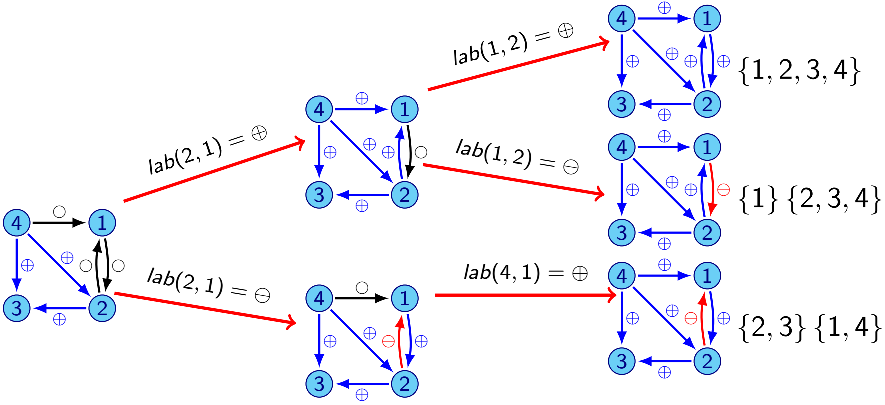

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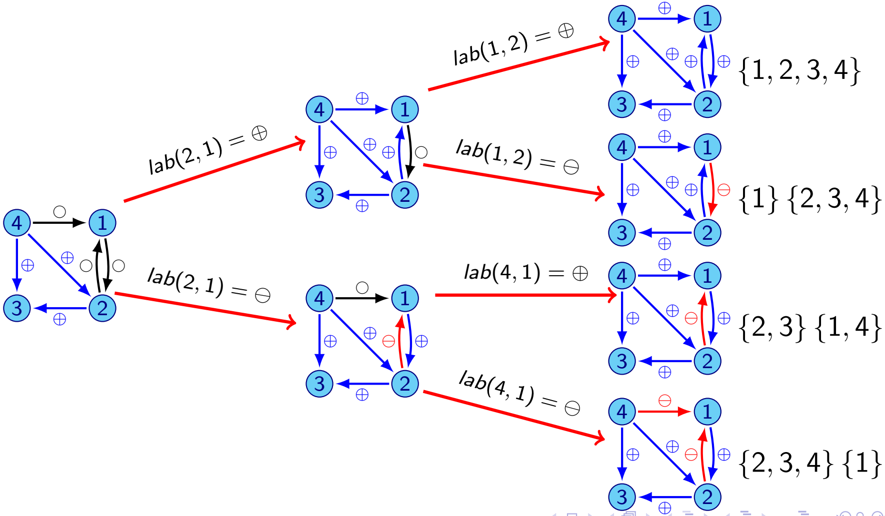


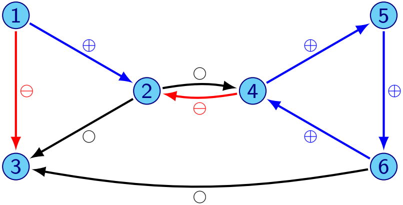
Reduce

Definition

Let (G, lab) be an update digraph and $\{G_1, \ldots, G_k\}$ its positive strongly connected components. We define its reduced labeled digraph by $R(G, lab) = (G_{\text{rd}} = (V_{\text{rd}}, A_{\text{rd}}), lab_{\text{rd}})$, where:

- $V_{\text{rd}} = \{v_1, \ldots, v_k\}$
- $A_{\text{rd}} = \{(v_i, v_j) | \exists (u, v) \in A(G) \cap \big(V(G_i) \times V(G_j)\big)\}$

$lab_{\text{rd}}(v_i, v_j) = lab(u, v)$, if there exists $(u, v) \in \big(V(G_i) \times V(G_j)\big) \cap \text{Sup}(lab)$ and $lab_{\text{rd}}(v_i, v_j) = \bigcirc$ otherwise



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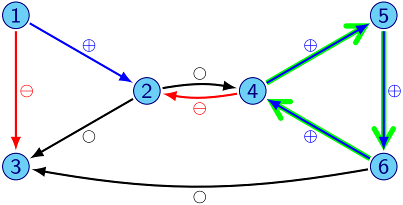

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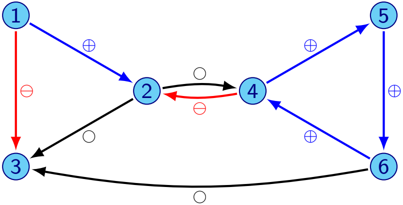
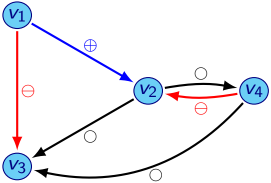
$lab_{\text{rd}}(v_i, v_j) = lab(u, v)$, if there exists $(u, v) \in \left(V(G_i) \times V(G_j)\right) \cap \text{Sup}(lab)$ and $lab_{\text{rd}}(v_i, v_j) = \bigcirc$ otherwise





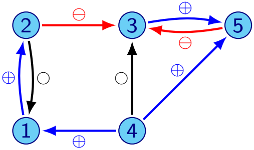
Divide and conquer

Divide by SCC

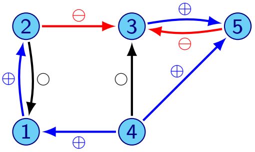
Let (G, lab) an update digraph with SCC $G_1, \dots, G_k$ (ordered) over its reverse extended digraph, then:

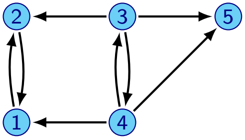
$$\mathcal{S}(G, lab) = \mathcal{S}\left(\widetilde{G}_1, lab|_{A(G_1)}\right) \circ_n \cdots \circ_n \mathcal{S}\left(\widetilde{G}_k, lab|_{A(G_k)}\right)$$

Example: Division by SCC

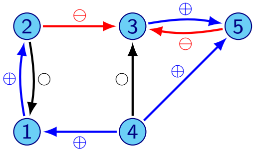


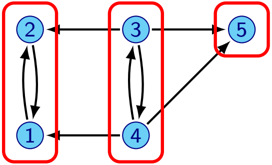
Example: Division by SCC



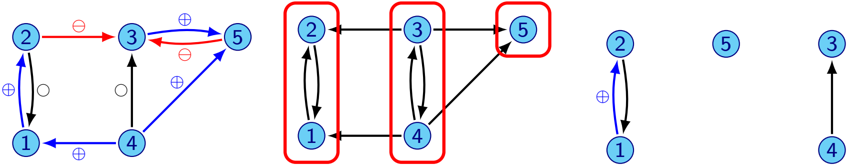


Example: Division by SCC

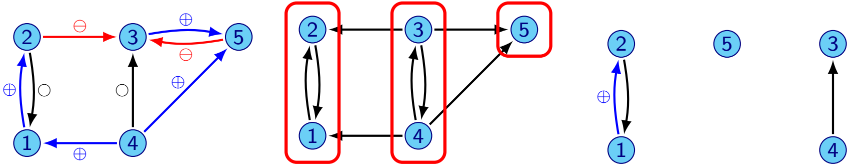




Example: Division by SCC

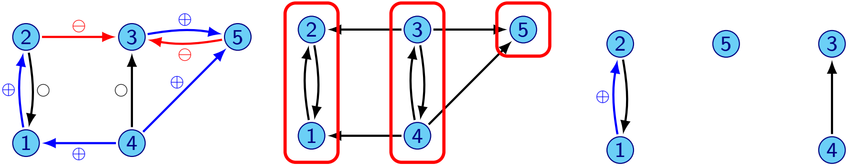


Example: Division by SCC



$$\mathcal{S}(G[1,2], lab) = \{\{1,2\},\{2\}\{1\}\}$$
$$\mathcal{S}(G[5], lab) = \{\{5\}\}$$
$$\mathcal{S}(G[3,4], lab) = \{\{3\}\{4\},\{4\}\{3\}\}$$

Example: Division by SCC



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Then,

$$\begin{aligned}\mathcal{S}(G, lab) &= \mathcal{S}(G[1,2], lab) \circ_n \mathcal{S}(G[5], lab) \circ_n \mathcal{S}(G[3,4], lab) \\ &= \{\{1,2\}\{5\}, \{1\}\{2\}\{5\}\} \circ_n \{\{3\}\{4\}, \{4\}\{3\}\} \\ &= \Big\{ \{1,2\}\{5\}\{3\}\{4\}, \{1\}\{2\}\{5\}\{3\}\{4\}, \\ &\quad \{1,2\}\{5\}\{4\}\{3\}, \{1\}\{2\}\{5\}\{4\}\{3\} \Big\}\end{aligned}$$

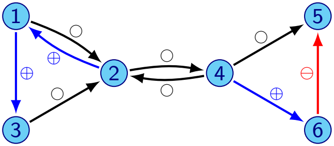
Divide and conquer

Divide by Bridges

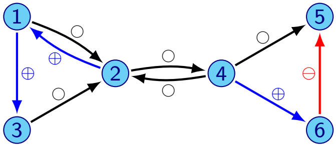
Let (G, lab) a connected digraph, G_U the underlying digraph of G and $uv \in E(G_U)$ a bridge that divide G in G_1 and G_2, then

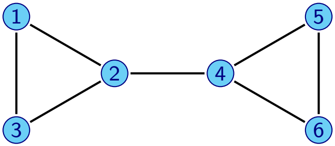
$$\mathcal{S}(G, lab) = \mathcal{S}(G_1, lab|_{A(G_1)}) \circ_{\{u,v\}} \mathcal{S}(G_2, lab|_{A(G_2)})$$

Example: Division by Bridges

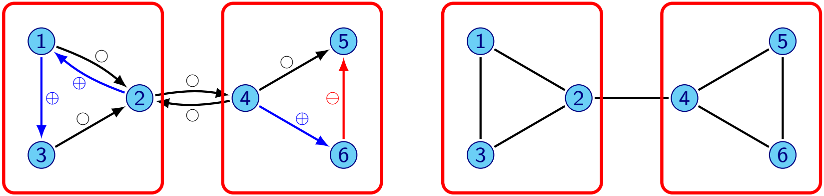


Example: Division by Bridges

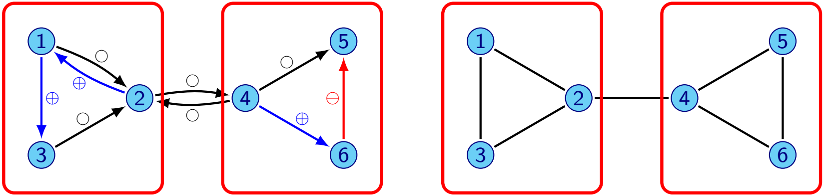




Example: Division by Bridges



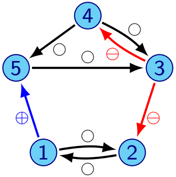
Example: Division by Bridges



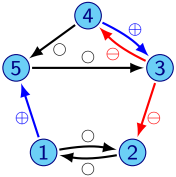
$$S_1 \circ_{nrm_{2,4}} S_2 = \{\{1,2,3\},\{3\}\{1,2\},\{3\}\{1\}\{2\}\} \circ_{nrm_{2,4}} \{\{6\}\{5\}\{4\},\{6\}\{4\}\{5\}\}$$

$\{1,2,3\}\{6\}\{5\}\{4\}$	$\{6\}\{5\}\{4\}\{1,2,3\}$	$\{6\}\{5\}\{4,1,2,3\}$
$\{3\}\{1,2\}\{6\}\{5\}\{4\}$	$\{6\}\{5\}\{4\}\{3\}\{1,2\}$	$\{3\}\{6\}\{5\}\{4,1,2\}$
$\{3\}\{1\}\{2\}\{6\}\{5\}\{4\}$	$\{6\}\{5\}\{4\}\{3\}\{1\}\{2\}$	$\{3\}\{1\}\{6\}\{5\}\{4,2\}$
$\{1,2,3\}\{6\}\{4\}\{5\}$	$\{6\}\{4\}\{5\}\{1,2,3\}$	$\{6\}\{4,1,2,3\}\{5\}$
$\{3\}\{1,2\}\{6\}\{4\}\{5\}$	$\{6\}\{4\}\{5\}\{3\}\{1,2\}$	$\{3\}\{6\}\{4,1,2\}\{5\}$
$\{3\}\{1\}\{2\}\{6\}\{4\}\{5\}$	$\{6\}\{4\}\{5\}\{3\}\{1\}\{2\}$	$\{3\}\{1\}\{6\}\{4,2\}\{5\}$

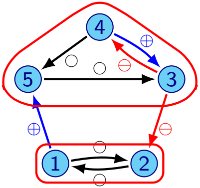
Example: UpdateLabel



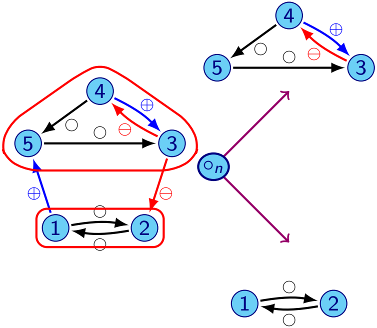
Example: UpdateLabel



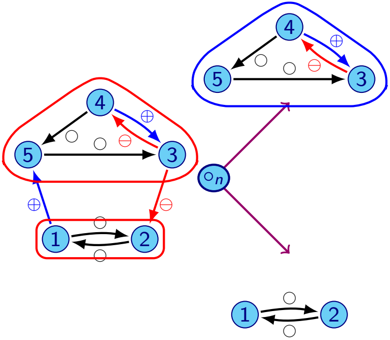
Example: UpdateLabel



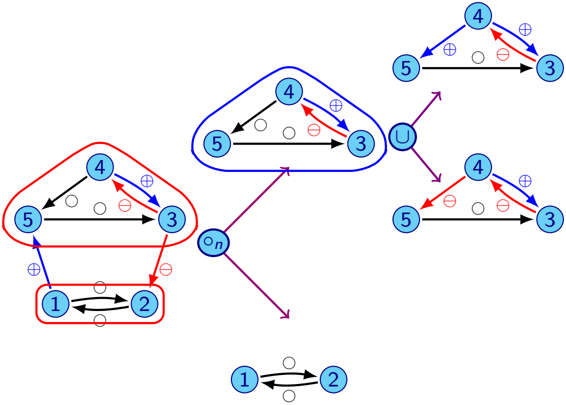
Example: UpdateLabel



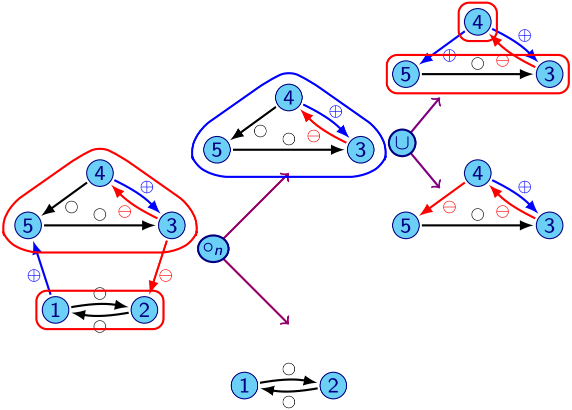
Example: UpdateLabel



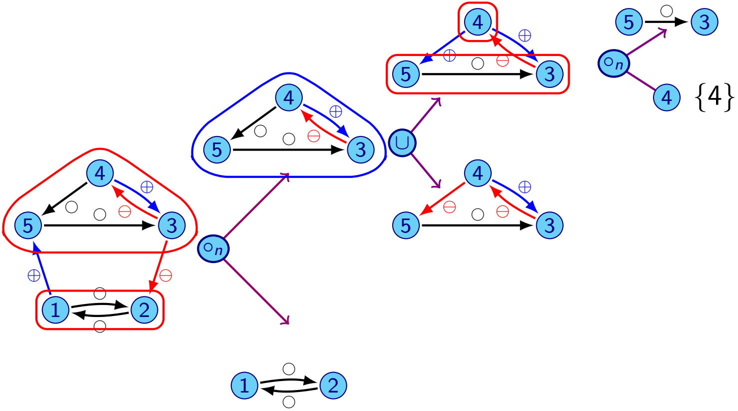
Example: UpdateLabel



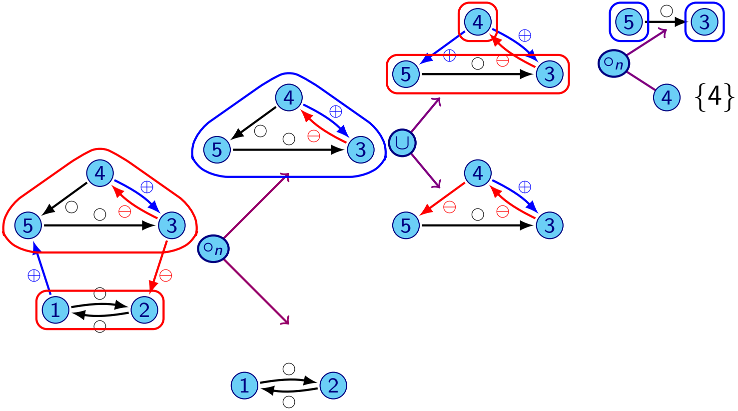
Example: UpdateLabel



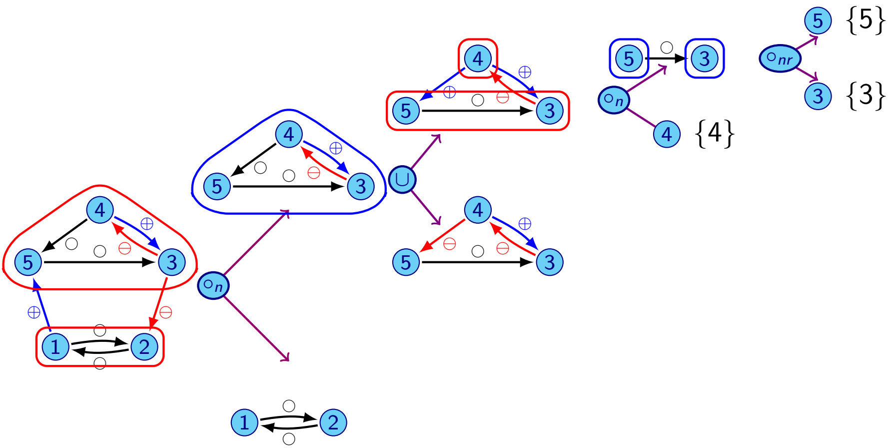
Example: UpdateLabel



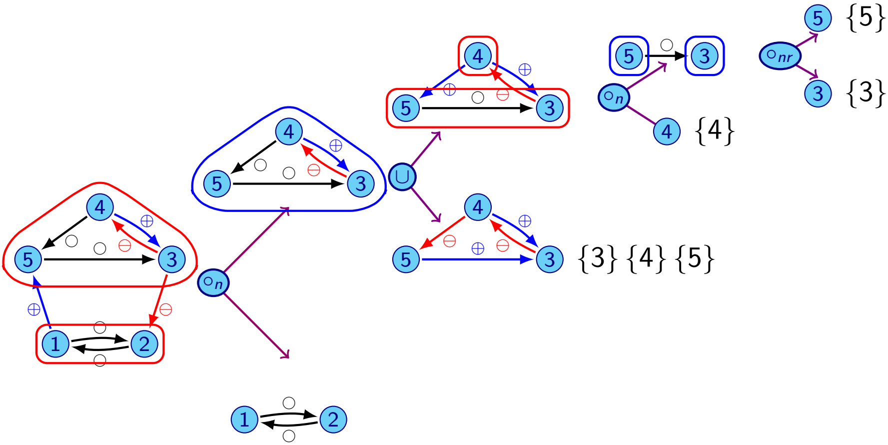
Example: UpdateLabel



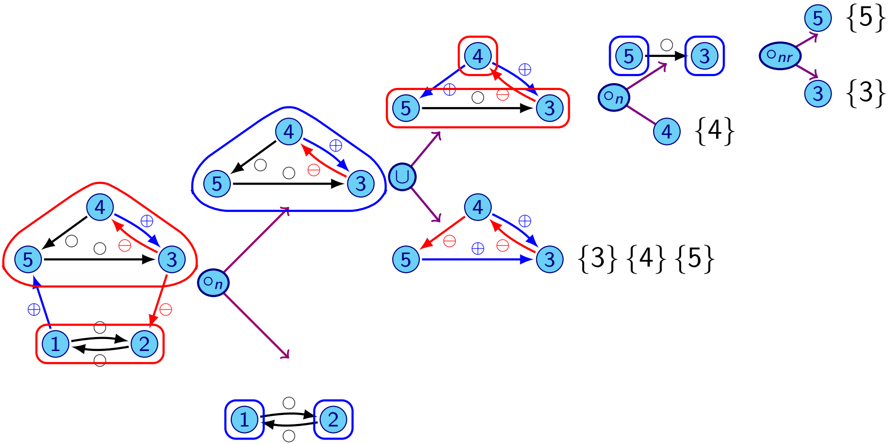
Example: UpdateLabel



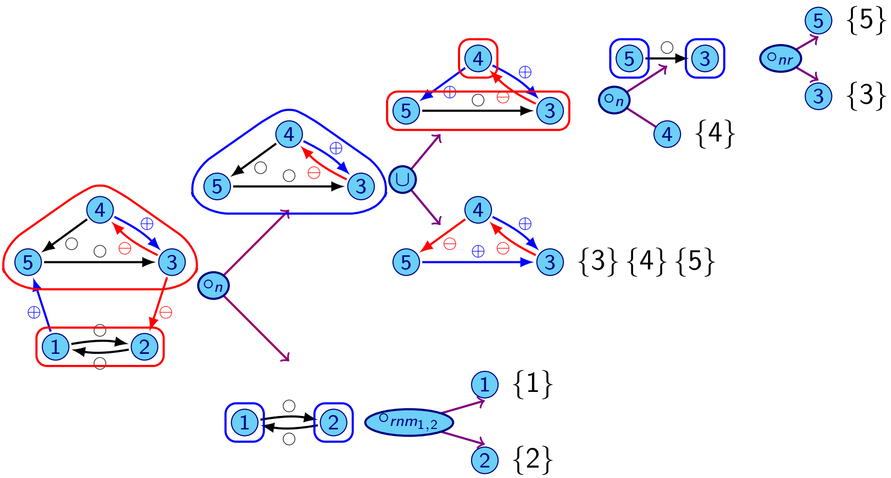
Example: UpdateLabel



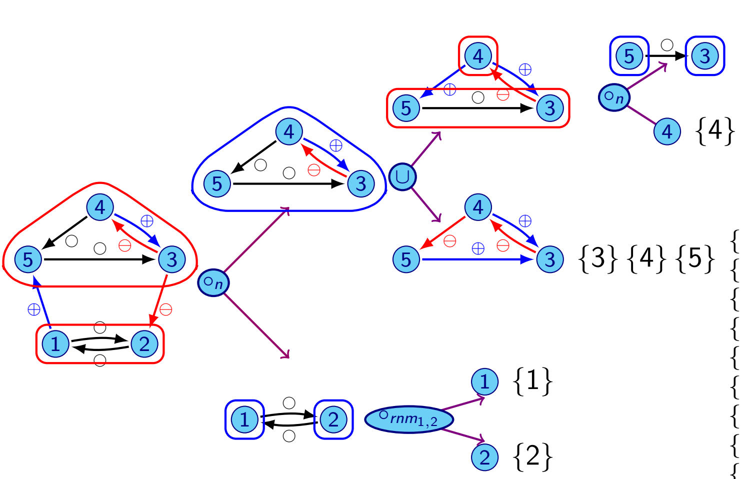
Example: UpdateLabel



Example: UpdateLabel



Example: UpdateLabel



{3} {5} {4} {1} {2}
{5} {3} {4} {1} {2}
{3} {4} {5} {1} {2}
{3} {5} {4} {2} {1}
{5} {3} {4} {2} {1}
{3} {4} {5} {2} {1}
{3} {5} {4} {1, 2}
{5} {3} {4} {1, 2}
{3} {4} {5} {1, 2}

Results

Graph	Nodes	Arcs	$\lvert\mathcal{S}(G, lab)\rvert$	SimpleLabel	Label
K_3	3	6	13	0.02	0.02
K_5	5	20	541	0.18	0.13
K_7	7	42	47 293	140.76	0.79
K_8	8	56	545 835	$>$ 22 200.00	1.90
P_{10}	10	18	19 683	4.43	0.02
P_{12}	12	22	177 147	313.68	0.02
P_{50}	50	98	3^{49}	–	0.09
P_{200}	200	398	3^{199}	–	0.17