Fixed points and disjoint cycles in Boolean networks

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Workshop

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A **boolean network** is a function

$$f : \{0,1\}^n \to \{0,1\}^n$$
$$x = (x_1, \dots, x_n) \mapsto f(x) = (f_1(x), \dots, f_n(x))$$

The **interaction graph** of f is the digraph G defined by

- the vertex set is $\{1, \dots, n\}$
- there is an arc $j \to i$ if f_i *depends on* x_j

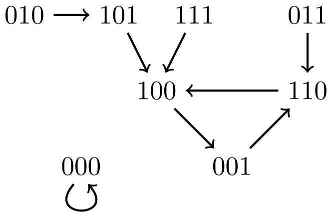
Example with $n = 3$

$$\begin{cases} f_1(x) &= x_2 \vee x_3 \\ f_2(x) &= \overline{x_1} \wedge \overline{x_3} \\ f_3(x) &= \overline{x_3} \wedge (x_1 \oplus x_2) \end{cases}$$

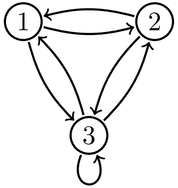
Table of f

x	$f(x)$
000	000
001	110
010	101
011	110
100	001
101	100
110	100
111	100

Dynamics of f



Interaction graph of f



Many applications

- Neural networks [McCulloch & Pitts 1943]
- **Gene networks** [Kauffman 1969, Tomas 1973]
- Epidemic diffusion, social network, etc

Very often

- reliable information concern the interaction graph
- fixed points have strong meaning

Natural question

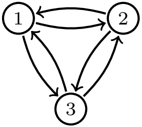
What can be said on the **dynamics** *of a boolean network according to its* **interaction graph** *only ?*

Graph parameters

$\boldsymbol{\phi(G)}$:= **maximum number of fixed points** in a boolean network with G as interaction graph

$\boldsymbol{\tau(G)}$:= **transversal number**
minimum Feedback Vertex Set (FVS)

$\boldsymbol{\nu(G)}$:= **packing number**
maximum number of vertex-disjoint cycles



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$\tau = 2$

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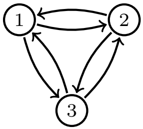

$\phi = 4$

$\tau = 2$

$\nu = 1$

Remark We always have $\nu \leq \tau$

Theorem [Riis 07, Aracena 08, Aracena-R-Salinas 16+]

For every digraph G,

$$\nu + 1 \le \phi \le 2^{\tau}$$

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Problem *Find a (reasonable) up. bound on ϕ that depends on ν only*

Theorem [Reed-Robertson-Seymour-Thomas 95]

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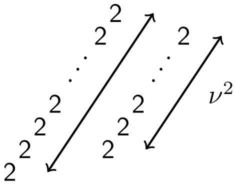

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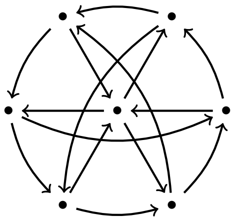

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Natural approach: try with particular classes of networks

$\phi_{\mathsf{ne}}(G)$:= **maximum number of fixed points** in a boolean network f with G as interaction graph and **Non-Expansive**:

$$d(f(x), f(y)) \leq d(x, y)$$

$\phi_{\mathsf{m}}(G)$:= **maximum number of fixed points** in a boolean network f with G as interaction graph and **Monotone**:

$$x \leq y \;\Rightarrow\; f(x) \leq f(y)$$

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$\nu = 2 \Rightarrow \phi_{\mathsf{m}} \leq 4$ tight and easy

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$\nu = 1 \Rightarrow \phi_{\mathsf{m}} \leq 2$ tight and easy

$\nu = 2 \Rightarrow \phi_{\mathsf{m}} \leq 4$ tight and easy

Problem $\nu = 3 \Rightarrow \phi_{\mathsf{m}} \leq ??$

¡Muchas gracias!